Assignment 8

1. Show that a metric space (X, d) is complete if and only if, whenever closed sets $E_k,\ k \geq 1$, satisfy $E_{k+1} \subset E_k$ and $\mathrm{diam}(E_k) \to 0$, $\bigcap_{k=1}^{\infty} E_k \neq \phi$. The definition of the diameter of a set in a metric space is self-evident. Provide one first.

2. Let $B(X, Y)$ denoted all bounded maps from metric spaces X to Y.

   (a) Show that the supnorm induces a complete metric on $B(X, Y)$ provided Y is complete.

   (b) Show that $C_b(X, Y) \subset B(X, Y)$ consisting of bounded, continuous maps is closed and hence complete in $B(X, Y)$.

3. Let (X, d) be a metric space. Fixing a point $p \in X$, for each x define a function
$$f_x(z) = d(z, x) - d(z, p).$$

   (a) Show that each f_x is a bounded, uniformly continuous function in X.

   (b) Show that the map $x \mapsto f_x$ is an isometric embedding of (X, d) to $C_b(X)$ (shorthand for $C_b(X, \mathbb{R})$) . In other words,
$$\|f_x - f_y\|_\infty = d(x, y), \quad \forall x, y \in X.$$

   (c) Deduce from (b) the completion theorem.

   This approach is much shorter than the proof given in notes. However, it is not so inspiring.

4. Let $f : E \to Y$ be a uniformly continuous map where $E \subset X$ and X, Y are metric spaces. Suppose that Y is complete. Show that there exists a uniformly continuous map F from $\overline{E}$ to Y satisfying $F = f$ in E. In other words, f can be extended to the closure of E preserving uniform continuity.

5. Let T be a continuous map on the complete metric space X. Suppose that for some k, T^k becomes a contraction. Show that T admits a unique fixed point. This generalizes the contraction mapping principle in the case $k = 1$.

6. Let K be a convex, closed and bounded set in $\mathbb{R}^n$ and $f : K \to K$ a C^1-map satisfying $|\partial f_i/\partial x_j| < 1$ in K. Show that f is a contraction and hence admits a unique fixed point.

7. Show that every continuous function from $[0, 1]$ to itself admits a fixed point. Here we don't need it a contraction. Suggestion: Consider the sign of $g(x) = f(x) - x$ at $0, 1$ where f is the given function.

8. Consider the iteration
$$x_{n+1} = \alpha x_n(1 - x_n),\ x_1 \in [0, 1]\ .$$
   Find

   (a) The range of α so that $\{x_n\}$ remains in $[0, 1]$.

   (b) The range of α so that the iteration has a unique fixed point 0 in $[0, 1]$.

   (c) Show that for $\alpha \in [0, 1]$ the fixed point 0 is attracting in the sense: $x_n \to 0$ whenever $x_0 \in [0, 1]$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0, f'(x_0) \neq 0$. Show that there exists some $\rho > 0$ such that
$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),$$
   is a contraction. This provides a justification for Newton's method in finding roots for an equation.